

FLOWS WITH LIMITED INTERSECTION OF WORLDLINES

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ABSTRACT. In this paper we define a flow with limited intersection of its worldlines and we construct and solve functional equations for such flow using a special kind of set embedding. For examples we use particular cases studied in the past by different authors. The connection to higher order ordinary differential equations is emphasized.

1. WORLDLINES OF THE FLOW

By the term *flow* we usually mean a one-parameter transformation group (see e.g. Chapter 1 in [1], 4 in [2] or section 1.3 v [3]), i.e. the action $\mathbb{R} \times M \ni (t, a) \mapsto ta \in M$ of the additive group of real numbers \mathbb{R} on some set M . The partial maps $\mathbb{R} \ni t \mapsto ta \in M$ defined for any $a \in M$ are called *worldlines* of the flow. It is clear that a flow is fully described by the set of its worldlines.

The term worldline is connected with preference to space-time approach in physics and is most used in relativity theory (e.g. Sections 7 to 9 in [4]). However its usage is not strictly connected to this area only. In the theory of flows the so-called *orbits* are intensively studied. Orbits are the sets $\cup_{t \in \mathbb{R}} \{ta\} \subset M$, i.e. the images of worldlines.

Rigorously, the original Minkovski term worldline [5] is not the map $\mathbb{R} \rightarrow M$ directly but its graph. In the case of worldlines $\mathbb{R} \ni t \mapsto ta \in M$ it means the set $\cup_{t \in \mathbb{R}} \{(t, ta)\} \subset \mathbb{R} \times M$. This should not be confusing since any map $\mathbb{R} \rightarrow M$ can be considered as a special case of a binary relation on $\mathbb{R} \times M$ (see e.g. 3.4 in [6] or Section 6 of Chapter 1 in [7]). Therefore it is possible to identify the map with its graph and

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hence, $\mathbb{R} \ni t \mapsto ta \in M$ and $\cup_{t \in \mathbb{R}} \{(t, ta)\} \subset \mathbb{R} \times M$ are just different descriptions of the same thing.

An important property of worldlines of one-parameter transformation group is the fact that they are disjoint. This is clear from the following reasoning. Two worldlines $\mathbb{R} \ni t \mapsto ta_1 \in M$ and $\mathbb{R} \ni t \mapsto ta_2 \in M$ are different if and only if there is $t_0 \in \mathbb{R}$ where the two worldlines as maps do not attain the same value, i.e. $t_0 a_1 \neq t_0 a_2$. Since we can cancel out the group element we successively obtain $a_1 \neq a_2$ and $ta_1 \neq ta_2$ for all $t \in \mathbb{R}$, so the worldlines are disjoint as graphs.

For completeness let us note that the orbit invariance with respect to the group action implies that the worldlines of the flow $\mathbb{R} \times M \ni (t, a) \mapsto ta \in M$ are at the same time the orbits of the flow $\mathbb{R} \times (\mathbb{R} \times M) \ni (s, (t, a)) \mapsto (s + t, (s + t)a) \in (\mathbb{R} \times M)$.

2. GENERALIZATION OF FLOWS

We usually talk about flows in case of dynamical systems with disjoint worldlines and each worldline can be uniquely determined by any one of its points. In this sense by flow we understand the map that assigns to a point of a worldline the whole worldline. Now we want to generalize the notion of the flow in case the worldlines are not disjoint anymore but the number of intersections of two different worldlines is further bounded above by a natural number denoted by k . In this new situation we can uniquely identify each worldline by its k different points. Now we can define the generalized flow as a map assigning to any k -point restriction of any worldline the whole worldline. In the sequel we formalize this construction.

Let \mathbb{R} denote the set of all real numbers, $I \subset \mathbb{R}$ its subset of cardinality $|I| > k$, M any general given set, M^I the set of all maps from I to M . We define the *set of worldlines with limited intersection* as the set $X \subset M^I$ such that for any $x_1, x_2 \in X$

$$(2.1) \quad |x_1 \cap x_2| \geq k \quad \Rightarrow \quad x_1 = x_2.$$

The cardinality of the intersection of maps $|x_1 \cap x_2|$ is defined here through identification of the maps with their graphs as described above. From now on, by worldlines we understand the elements of the set X . By the *set of all k -restrictions of the worldline* we mean the set $C_k(x)$ of all k -combinations, i.e. the set of all subsets $a \subset x$ such that $|a| = k$. Since x is a map then also any combination $a \in C_k(x)$ is a map and satisfies $\text{dom } a \in C_k(I)$, $a = x|_{\text{dom } a}$. We will call the set of all k -restrictions of all worldlines $A = \cup_{x \in X} C_k(x) \subset \cup_{\alpha \in C_k(I)} M^\alpha$ shortly as the *set of restrictions*.

We consider now the binary relation $\varphi \subset A \times X$ defined by $(a, x) \in \varphi \Leftrightarrow a \subset x$. Since $x \in X \subset M^I$ implies $|x| = |I| > k$, we have $C_k(x) \neq \emptyset$. Hence, there exists $a \in A$ such that $(a, x) \in \varphi$ and therefore φ is surjective. By definition of A , φ is also co-surjective, i.e. for all $a \in A$ there is $x \in X$ such that $(a, x) \in \varphi$. Moreover, such x is unique since $(a, x_1) \in \varphi \wedge (a, x_2) \in \varphi$ yields $a \in C_k(x_1) \cap C_k(x_2) = C_k(x_1 \cap x_2)$, therefore $|x_1 \cap x_2| \geq k$ and by (2.1) we get $x_1 = x_2$. It follows that the relation φ is a map with domain $\text{dom } \varphi = A$ and codomain $\text{cod } \varphi = X$.

This allows to define the *flow with limited intersection of worldlines* as such a surjective map $\varphi: A \ni a \mapsto \varphi_a \in X$, where $A = \cup_{x \in X} C_k(x)$ is the set of restrictions, $X \subset M^I$ is the set of worldlines satisfying the condition (2.1), that $\varphi_a \supset a$ holds for all $a \in A$.

3. FUNCTIONAL EQUATIONS

Theorem 1. *The surjective map $\varphi: a \mapsto \varphi_a$, $\text{dom } \varphi \subset \cup_{\alpha \in C_k(I)} M^\alpha$, $\text{cod } \varphi \subset M^I$, is a flow with limited intersection of worldlines if and only if for any $a \in \text{dom } \varphi$, $\beta \in C_k(I)$*

$$(3.1) \quad \varphi_{\varphi_a|_\beta} = \varphi_a, \quad \varphi_a|_{\text{dom } a} = a.$$

Proof: Let φ be a flow with limited intersection of worldlines and let $a \in \text{dom } \varphi$, $\beta \in C_k(I)$. Since $\text{dom } \varphi_a = I$ we have $\varphi_a|_\beta \in C_k(\varphi_a) \subset \text{dom } \varphi$ and by the definition of flow $\varphi_{\varphi_a|_\beta} \supset \varphi_a|_\beta$. Because $\varphi_a \supset \varphi_a|_\beta$ we obtain $|\varphi_{\varphi_a|_\beta} \cap \varphi_a| \geq |\varphi_a|_\beta| = |\beta| = k$ and using (2.1) we obtain $\varphi_{\varphi_a|_\beta} = \varphi_a$. Furthermore, the definition of the flow guarantees also $\varphi_a \supset a$ and hence, $\varphi_a|_{\text{dom } a} \supset a|_{\text{dom } a} = a$. But $|\varphi_a|_{\text{dom } a}| = |a| = k$ which implies $\varphi_a|_{\text{dom } a} = a$. Therefore, the equalities (3.1) are verified and the proof in one direction is finished.

Let now φ be surjective with $\text{dom } \varphi \subset \cup_{\alpha \in C_k(I)} M^\alpha$ and $\text{cod } \varphi \subset M^I$ satisfying conditions (3.1) for all $a \in \text{dom } \varphi$ and $\beta \in C_k(I)$. Then $\varphi_a|_{\text{dom } a} = a$ implies $\varphi_a \supset a$.

Let $x_1, x_2 \in \text{cod } \varphi$ satisfy $|x_1 \cap x_2| \geq k$. By surjectivity there exist $b, c \in \text{dom } \varphi$ such that $x_1 = \varphi_b$, $x_2 = \varphi_c$. The fact that intersection of maps is a map again entails $|\text{dom}(\varphi_b \cap \varphi_c)| = |\varphi_b \cap \varphi_c| \geq k$, therefore there exists $\delta \in C_k(\text{dom}(\varphi_b \cap \varphi_c))$. Then for such δ we have $\varphi_b|_\delta = \varphi_c|_\delta$. By (3.1), $x_1 = \varphi_b = \varphi_{\varphi_b|_\delta} = \varphi_{\varphi_c|_\delta} = \varphi_c = x_2$. This proves condition (2.1).

Assume that $b \in \cup_{x \in \text{cod } \varphi} C_k(x)$. Then there is $x_0 \in \text{cod } \varphi$ such that $b \in C_k(x_0)$. By surjectivity, there exist $a \in \text{dom } \varphi$ such that $x_0 = \varphi_a$. We use the notation $\beta = \text{dom } b$. Then, $\beta \in C_k(\text{dom } \varphi_a) = C_k(I)$. By (3.1), we also have $\varphi_{\varphi_a|_\beta} = \varphi_a$, hence $b = \varphi_a|_\beta \in \text{dom } \varphi$. This shows

$\cup_{x \in \text{cod } \varphi} C_k(x) \subset \text{dom } \varphi$. Suppose on the other hand that $a \in \text{dom } \varphi$. Then $|a| = k$, $a \subset \varphi_a$ and therefore $a \in C_k(\varphi_a) \subset \cup_{x \in \text{cod } \varphi} C_k(x)$. This proves $\text{dom } \varphi = \cup_{x \in \text{cod } \varphi} C_k(x)$.

Because all the requirements of the definition are fulfilled, the map φ is a flow with limited intersection of worldlines. This completes the proof in both directions. \square

4. EXAMPLES

Flow without intersection of worldlines: By (2.1), a flow without intersection of worldlines is a flow with $k = 1$. Usually, we assume the set of restrictions in the form $\text{dom } \varphi = \cup_{\alpha \in C_1(\mathbb{R})} M^\alpha$. Among others this also means that $a \in \text{dom } \varphi$, $a = \{(r, m)\}$ where $(r, m) \in \mathbb{R} \times M$ and $\text{cod } \varphi \subset M^{\mathbb{R}}$. We can rewrite the functional equation (3.1) for nonautonomous flow without intersection of worldlines as

$$(4.1) \quad F(t, r, F(r, s, m)) = F(t, s, m), \quad F(s, s, m) = m,$$

where $F: \mathbb{R} \times \mathbb{R} \times M \rightarrow M$ is a map with three variables and the connection to the flow stands $F(t, s, m) = \varphi_{\{(s, m)\}}(t)$.

The functional equations (4.1) are the generalization to 8.1.3(8)[8] of the Sincov's equation (D. M. Sincov 1903). It was already studied by many authors (A. N. Kolmogorov 1931, M. Fréchet 1932, J. Aczél 1955, M. Hosszú 1958 and others) and is used, i.a., in the theory of stochastic processes. Its detailed description and references to the original work can be found in 8.1.3 and 8.1.4 of the Aczél's book [8].

Autonomous flow: The most familiar flow without intersection of worldlines is the autonomous flow. The flow autonomy means the closure property of $\text{cod } \varphi$ under all translation $\tau_s: \mathbb{R} \ni t \mapsto t + s \in \mathbb{R}$, i.e. $x \in \text{cod } \varphi \Rightarrow x \circ \tau_s \in \text{cod } \varphi$ for all $s \in \mathbb{R}$. By (3.1), we know that $\varphi_{\{(r, m)\}}(s) = \varphi_{\{(r, m)\}} \circ \tau_s(0) = \varphi_{\varphi_{\{(r, m)\}} \circ \tau_s|_{\{r-s\}}}(0) = \varphi_{\varphi_{\{(r, m)\}}|_{\{r\}} \circ \tau_s}(0) = \varphi_{\varphi_{\{(r, m)\}}|_{\text{dom}\{\{(r, m)\}} \circ \tau_s}}(0) = \varphi_{\{(r, m)\} \circ \tau_s}(0) = \varphi_{\{(r-s, m)\}}(0)$, where \circ means the composition of binary relations. Hence, we can rewrite the functional equation (3.1) of autonomous flow without intersection of worldlines in the form

$$F(r, F(s, m)) = F(r + s, m), \quad F(0, m) = m,$$

where $F: \mathbb{R} \times M \rightarrow M$ is map of two variables and its relation to the flow are the following equalities $F(r, m) = \varphi_{\{(r, m)\}}(0)$ and $\varphi_{\{(r, m)\}}(s) = F(r - s, m)$.

The functional equations of autonomous flow without intersection of worldlines describe a one-parametric group of transformations $M \rightarrow M$ and are used under the name *translation equations* in iteration theory

and dynamical systems and in differentiable case in the theory of Lie groups and first order differential equations (see e.g. (3), (17) in [9] and (1.9), (1.10) in [3]). The general solution is described in Section 8.2.2. of [8] and this book contains also references to the relevant original works from the middle of the last century.

The set of solutions of the Dirichlet problem in ODE's: We assume an ordinary differential equation $\ddot{x} = f(t, x, \dot{x})$ that has for any Dirichlet boundary condition $x(\alpha) = a, x(\beta) = b, \alpha \neq \beta$ a unique solution $x: \mathbb{R} \rightarrow \mathbb{R}^n$ satisfying this condition. The solutions of all Dirichlet problems corresponding to such a differential equation form a set of worldlines with limited intersections. Actually, the worldlines that intersect at least twice satisfy the same Dirichlet condition and hence they coincide. Thus, in this case $k = 2, I = \mathbb{R}$ and $M = \mathbb{R}^n$. This situation is described by P. Chládek in [10] by using the equations

$$(4.2) \quad F(\tau, \alpha, \beta, a, b) = F(\tau, \gamma, \delta, F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)),$$

$$(4.3) \quad F(\alpha, \alpha, \beta, a, b) = a, \quad F(\beta, \alpha, \beta, a, b) = b,$$

where $F: \mathbb{R} \times P_2(\mathbb{R}) \times (\mathbb{R}^n)^2 \rightarrow \mathbb{R}^n$ is a function and $P_2(\mathbb{R}) \subset \mathbb{R}^2$ the set of all 2-permutations (without repetition) of elements of \mathbb{R} . It is a special case of functional equations (3.1) where the unknown function F is related to the flow $\varphi: \cup_{\sigma \in C_2(\mathbb{R})} (\mathbb{R}^n)^\sigma \rightarrow \text{cod } \varphi \subset (\mathbb{R}^n)^\mathbb{R}$ as follows:

$$(4.4) \quad F(\tau, \alpha, \beta, a, b) = \varphi_{\{(\alpha, a), (\beta, b)\}}(\tau).$$

Harmonic oscillator: A concrete case of the previous example for $n = 1$ and $k = 2$ is the differential equation $\ddot{x} = -x$ describing the behavior of a harmonic oscillator. Since any of its maximal solutions has period 2π and any two maximal solutions intersect, they must intersect infinitely many times. Hence, they are not worldlines with limited intersection. But we can consider the set X as the set of all solutions defined on an open interval I of length at most π . Then the condition (2.1) is satisfied. To be exact, we fix $I = (-\frac{\pi}{2}, +\frac{\pi}{2})$. Then $A = \text{dom } \varphi = \cup_{\alpha \in C_2(I)} \mathbb{R}^\alpha$, $X = \text{cod } \varphi \subset \mathbb{R}^I$ and the equation $\varphi_a|_{\text{dom } a} = a$ admits for any $a \in \text{dom } \varphi$ exactly one solution φ_a given by

$$\varphi_a(t) = \sum_{(i,j) \in P_2(\text{dom } a)} a_i \frac{\sin(t-j)}{\sin(i-j)}.$$

The relation $\varphi_{\varphi_a|_\beta} = \varphi_a$ becomes a goniometrical identity where for any $t \in I$, $a \in A$ and $\beta \in C_2(I)$ we have

$$\sum_{(r,s) \in P_2(\beta)} \sum_{(i,j) \in P_2(\text{dom } a)} a_i \frac{\sin(r-j) \sin(t-s)}{\sin(i-j) \sin(r-s)} = \sum_{(i,j) \in P_2(\text{dom } a)} a_i \frac{\sin(t-j)}{\sin(i-j)}.$$

Varying the interval I the flow with limited intersection of worldlines allows us to map step by step completely all the solutions of the equation $\ddot{x} = -x$. In the last section of this paper it is proven that similar process is possible for a very wide class of ordinary differential equations.

Lagrange interpolating polynomials: An interesting example of a set of worldlines with limited intersection for a general natural number k is a set of polynomials. Let $X = \text{cod } \varphi \subset \mathbb{R}^{\mathbb{R}}$ is the set of all polynomials of degree less than k . The number $|x_1 \cap x_2|$ for $x_1, x_2 \in X$ is the cardinality of the set of real roots of the polynomial $x_1 - x_2$. There are at least k such roots if and only if we deal here with the zero polynomial, i.e. $x_1 = x_2$. The condition (2.1) is therefore satisfied. The set of restrictions is $A = \text{dom } \varphi = \cup_{\alpha \in C_k(\mathbb{R})} \mathbb{R}^\alpha$ in this case. The equation $\varphi_a|_{\text{dom } a} = a$ has for any $a \in \text{dom } \varphi$ exactly one solution φ_a , namely the *Lagrange interpolation formula* (see e. g. (2.5.3) in [11]) given by

$$\varphi_a(t) = \sum_{i \in \text{dom } a} a_i \prod_{j \in \text{dom } a \setminus \{i\}} \frac{t-j}{i-j}.$$

The equation $\varphi_{\varphi_a|_\beta} = \varphi_a$ is then the summation identity for the Lagrange interpolation formula, where for any $t \in \mathbb{R}$, $a \in A$, $\beta \in C_k(\mathbb{R})$ we have

$$\sum_{r \in \beta} \sum_{i \in \text{dom } a} a_i \prod_{j \in \text{dom } a \setminus \{i\}} \frac{r-j}{i-j} \prod_{s \in \beta \setminus \{r\}} \frac{t-s}{r-s} = \sum_{i \in \text{dom } a} a_i \prod_{j \in \text{dom } a \setminus \{i\}} \frac{t-j}{i-j}.$$

It is easy to see that the summation identity is equivalent to the so called *Cauchy relations* (2.5.13) [11].

5. FUNCTIONAL EQUATION SOLUTIONS

In the sequel we use the term *k-frontal embedding* of a set W into the Cartesian power M^q with $q > k$ and by this we mean an embedding with the property that any Cartesian projection of the set W into each M^k is a bijection. Recalling the ordering of the set $I \subset \mathbb{R}$ we can consider the sets M^I and for $\alpha \subset I$ also M^α as Cartesian powers. Then

by a k -frontal embedding into M^I we understand a surjection ω with $\text{cod } \omega \subset M^I$ such that any surjection

$$(5.1) \quad \omega_\beta: \text{dom } \omega \ni w \mapsto \omega(w)|_\beta \in \text{cod } \omega_\beta \subset M^\beta,$$

where $\beta \in C_k(I)$, is a bijection at the same time. The significance of this definition for the solution of functional equations of a flow is clear from the following theorem.

Theorem 2. *If ω is a k -frontal embedding into M^I then*

$$(5.2) \quad \varphi: \bigcup_{\sigma \in C_k(I)} \text{cod } \omega_\sigma \ni a \mapsto \omega \circ \omega_{\text{dom } a}^{-1}(a) \in \text{cod } \omega$$

is a solution of functional equations (3.1).

On the other hand, if the surjective map φ with $\text{dom } \varphi \subset \bigcup_{\alpha \in C_k(I)} M^\alpha$ and $\text{cod } \varphi \subset M^I$ is a solution of functional equations (3.1) then there exists a k -frontal embedding into M^I ω satisfying (5.2).

Proof: Suppose that ω is a k -frontal embedding into M^I and φ is a map given by (5.2). Then for any $a \in \text{dom } \varphi$ and for any $\beta \in C_k(I)$ the next equalities hold $\varphi_a|_\beta = (\omega \circ \omega_{\text{dom } a}^{-1}(a))|_\beta = \omega(\omega_{\text{dom } a}^{-1}(a))|_\beta = \omega_\beta \circ \omega_{\text{dom } a}^{-1}(a)$. Since $\text{dom } \varphi_a|_\beta = \beta$ and thus $\varphi_a|_\beta \in \text{cod } \omega_\beta$, we get $\varphi_{\varphi_a|_\beta} = \omega \circ \omega_\beta^{-1}(\omega_\beta \circ \omega_{\text{dom } a}^{-1}(a)) = \omega \circ \omega_{\text{dom } a}^{-1}(a) = \varphi_a$, $\varphi_a|_{\text{dom } a} = \omega(\omega_{\text{dom } a}^{-1}(a))|_{\text{dom } a} = \omega_{\text{dom } a}(\omega_{\text{dom } a}^{-1}(a)) = a$, therefore φ is a solution of equations (3.1).

Assume on the other side that φ with $\text{dom } \varphi \subset \bigcup_{\alpha \in C_k(I)} M^\alpha$ and $\text{cod } \varphi \subset M^I$ is a solution of equations (3.1). Let $\omega = \text{id}_{\text{cod } \varphi}$. Then ω is a surjection and $\text{cod } \omega = \text{cod } \varphi \subset M^I$. Consider $\beta \in C_k(I)$ and two elements $w_1, w_2 \in \text{dom } \omega$ such that $\omega_\beta(w_1) = \omega_\beta(w_2)$. Thanks to surjectivity of the map φ there exist $a_1, a_2 \in \text{dom } \varphi$ such that $\varphi_{a_1} = w_1$ and $\varphi_{a_2} = w_2$. Since $\varphi_{a_1}|_\beta = \omega_\beta(w_1) = \omega_\beta(w_2) = \varphi_{a_2}|_\beta$, by (3.1) we also have $w_1 = \varphi_{a_1} = \varphi_{\varphi_{a_1}|_\beta} = \varphi_{\varphi_{a_2}|_\beta} = \varphi_{a_2} = w_2$. This shows the injectivity of surjective maps (5.1) and therefore, their bijectivity. Hence, ω is a k -frontal embedding into M^I .

Since $\text{cod } \omega_\sigma = \bigcup_{x \in \text{cod } \varphi} \{x|_\sigma\}$ and since $\bigcup_{\sigma \in C_k(I)} \{x|_\sigma\} = C_k(x)$, then $\bigcup_{\sigma \in C_k(I)} \text{cod } \omega_\sigma = \bigcup_{x \in \text{cod } \varphi} C_k(x)$ and by Theorem 1 and by definition of a flow with limited intersection of worldlines it follows that $\text{dom } \varphi = \bigcup_{\sigma \in C_k(I)} \text{cod } \omega_\sigma$.

If we consider now $a \in \bigcup_{\sigma \in C_k(I)} \text{cod } \omega_\sigma$, by (3.1), it holds $\text{cod } \omega \ni \varphi_a = \omega_{\text{dom } a}^{-1}(\omega_{\text{dom } a}(\varphi_a)) = \omega_{\text{dom } a}^{-1}(\varphi_a|_{\text{dom } a}) = \omega_{\text{dom } a}^{-1}(a) = \omega \circ \omega_{\text{dom } a}^{-1}(a)$, thus ω satisfies (5.2). \square

Solution for $k = 1$: We solve the equations (4.1). The embedding $\omega: M \rightarrow \text{cod } \omega \subset M^I$ can be expressed by the system $\{G_t\}_{t \in I}$ of maps $G_t: M \ni w \mapsto \omega(w)(t) \in M$. To obtain a 1-frontal embedding ω of

M into M^I we need all the maps (5.1) to be bijections. This can be achieved only if every map of the system $\{G_t\}_{t \in I}$ is a bijection. Indeed, it is sufficient to take any system $\{G_t\}_{t \in I}$ of bijections $G_t: M \rightarrow M$ and according to the Theorem 2 the map

$$F: \mathbb{R} \times \mathbb{R} \times M \ni (t, s, m) \mapsto F(t, s, m) = \varphi_{\{(s, m)\}}(t) = \omega(\omega_{\{s\}}^{-1}(\{(s, m)\}))(t) = G_t(\omega_{\{s\}}^{-1}(\{(s, m)\})) = G_t(G_s^{-1}(m)) \in M$$

is a solution of equations (4.1). This solution is known already for some time and is described by Theorem 2 in Section 8.1.3 of the book [8].

Solution for $k = 2$: We can also solve the equations (4.2), (4.3) thanks to a 2-frontal embedding. We express the embedding $\omega: \mathbb{R}^{2n} \rightarrow \text{cod } \omega \subset (\mathbb{R}^n)^\mathbb{R}$ using maps of Cartesian spaces $G: \mathbb{R} \times \mathbb{R}^{2n} \ni (\tau, w) \mapsto \omega(w)(\tau) \in \mathbb{R}^n$. Should ω be a 2-frontal embedding into $(\mathbb{R}^n)^\mathbb{R}$, the equation

$$(5.3) \quad G(\alpha, w) = a, \quad G(\beta, w) = b$$

must have for any $(\alpha, \beta) \in P_2(\mathbb{R})$, $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ exactly one solution for $w \in \mathbb{R}^{2n}$. If we find a map G with the properties above then at the same time we have solution of equations (4.2) and (4.3). Indeed, by (4.4), (5.1) and (5.2) it holds $F(\tau, \alpha, \beta, a, b) = G(\tau, H(\alpha, \beta, a, b))$ providing that $H: P_2(\mathbb{R}) \times (\mathbb{R}^n)^2 \rightarrow \mathbb{R}^{2n}$ is a map that assigns to its arguments the solution

$$(5.4) \quad w = H(\alpha, \beta, a, b)$$

of equations (5.3). In fact, eliminating a and b from (5.3) and (5.4), we get $w = H(\alpha, \beta, G(\alpha, w), G(\beta, w))$ which implies (4.2). On the other side, eliminating w we get $G(\alpha, H(\alpha, \beta, a, b)) = a$, $G(\beta, H(\alpha, \beta, a, b)) = b$, which implies (4.3). As far as the authors of this paper are aware such general solution of Chládek's equations described herein are presented for the first time.

6. THE SMOOTH CASE

For simplicity we restrict our attention to maps with continuous partial derivatives of all orders and we call them *smooth*. In the previous section we transferred the problem of solving functional equations for a flow with limited intersection of worldlines to the problem of finding the corresponding k -frontal embedding. This arises the question of how difficult it is to find such an embedding. This section shows that it is very easy in the smooth case. The following Lemma gives sufficient condition for local generating of k -frontal embedding by some smooth map.

Lemma. Let n and k be natural numbers, G a smooth map, $\text{dom } G \subset \mathbb{R} \times (\mathbb{R}^n)^k$ an open set, $\text{cod } G = \mathbb{R}^n$, $J: \text{dom } G \rightarrow \mathbb{R}$ the Jacobian of the map

$$\text{dom } G \ni (t, w) \rightarrow \left(t, G(t, w), \frac{\partial G(t, w)}{\partial t}, \dots, \frac{\partial^{k-1} G(t, w)}{\partial t^{k-1}} \right) \in \mathbb{R} \times (\mathbb{R}^n)^k$$

and let $(t_0, w_0) \in \text{dom } G$ be a point where $J(t_0, w_0) \neq 0$. Then there exists such an open interval $I \subset \mathbb{R}$ and such an open set $U \subset (\mathbb{R}^n)^k$ that $(t_0, w_0) \in I \times U \subset \text{dom } G$ and the surjective map

$$(6.1) \quad \omega: U \ni w \mapsto (I \ni t \mapsto G(t, w) \in \mathbb{R}^n) \in \text{cod } \omega$$

is a k -frontal embedding of U into $(\mathbb{R}^n)^I$.

Proof: Since $\text{dom } G$ is an open set there is such an open interval $I \subset \mathbb{R}$ and such an open set $U \subset (\mathbb{R}^n)^k$ that $(t_0, w_0) \in I \times U \subset \text{dom } G$. We construct a map $K: I^k \times U \rightarrow (\mathbb{R}^n)^k$ assigning to any $(\tau, w) \in I^k \times U$ the value

$$K(\tau, w) = (K_1(t_1, w), K_2(t_1, t_2, w), \dots, K_k(t_1, t_2, \dots, t_k, w)),$$

where $\tau = (t_1, t_2, \dots, t_k)$, $K_1(t_1, w) = G(t_1, w)$ and K_2, \dots, K_k are recurrently given by

$$K_{i+1}(t_1, t_2, \dots, t_{i+1}, w) = \int_0^1 K'_i(t_1(1-s) + t_2 s, t_3, \dots, t_{i+1}, w) ds,$$

where K'_i denotes the partial derivative of K_i with respect to the first variable. A direct calculation of the integral on the right side easily verifies that the smoothness of K_i implies the smoothness of K_{i+1} . The map $K_1 = G|_{I \times U}$ is smooth by the assumptions of the Lemma, hence the map K as constructed above is also smooth. In the point (τ_0, w_0) , where $\tau_0 = (t_0, t_0, \dots, t_0) \in I^k$, the Jacobian of the smooth map $(\tau, w) \mapsto (\tau, K(\tau, w))$ is equal to $J(t_0, w_0)$ and therefore, by assumptions of the Lemma is nonzero.

Using the inverse function theorem (see e.g. Theorem 1A.1 in [12]) we can shrink the open interval I and the open set U preserving the condition $(t_0, w_0) \in I \times U \subset \text{dom } G$ in such a way that $(\tau, w) \mapsto (\tau, K(\tau, w))$ is an injection. But then also the map $w \mapsto K(\tau, w)$ is an injection for any $\tau \in I^k$. We consider $\beta \in C_k(I)$ and a k -permutation without repetition $(t_1, t_2, \dots, t_k) \in P_k(\beta) \subset \beta^k \subset I^k$. Since, for $i = 1, 2, \dots, k$ we have

$$K_i(t_1, t_2, \dots, t_i, w) = \sum_{j \in \{1, 2, \dots, i\}} G(t_j, w) \prod_{m \in \{1, 2, \dots, i\} \setminus \{j\}} \frac{1}{t_j - t_m},$$

the map $w \mapsto (G(t_1, w), G(t_2, w), \dots, G(t_k, w))$ is an injection too. Finally, by (5.1) and (6.1), for any $\beta \in C_k(I)$ the surjection

$$\omega_\beta: w \mapsto \{(t_1, G(t_1, w)), (t_2, G(t_2, w)), \dots, (t_k, G(t_k, w))\}$$

is also an injection, i.e. ω_β is a bijection. By definition of k -frontal embedding the map (6.1) is a k -frontal embedding of U into $(\mathbb{R}^n)^I$ and the proof is complete. \square

7. ORDINARY DIFFERENTIAL EQUATIONS

Now we formulate and prove a theorem on local description of the set of solutions of a differential equation using a flow with limited intersection of worldlines. Consider an ordinary differential equation

$$(7.1) \quad x^{(k)}(t) = f(t, x^{(0)}(t), x^{(1)}(t), \dots, x^{(k-1)}(t)),$$

where t is an independent variable, $x^{(i)}$ are derivatives of the unknown function x , especially $x^{(0)} = x$ and $x^{(i+1)}(t) = dx^{(i)}(t)/dt$, with Cauchy conditions

$$(7.2) \quad (x^{(0)}(t_0), x^{(1)}(t_0), \dots, x^{(k-1)}(t_0)) = w.$$

Theorem 3. *Let n and k be natural numbers, let f be a smooth map where $\text{dom } f \subset \mathbb{R} \times (\mathbb{R}^n)^k$ is an open set and $\text{cod } f = \mathbb{R}^n$ and let $(t_0, w_0) \in \text{dom } f$. Then there exists an open interval $I \subset \mathbb{R}$ and an open set $U \subset (\mathbb{R}^n)^k$ such that $(t_0, w_0) \in I \times U \subset \text{dom } f$ and the set X of all solutions x of the ordinary differential equation (7.1) defined on I and satisfying the Cauchy conditions (7.2) with $w \in U$ is a set of worldlines with limited intersection.*

Proof: We make use of the Lemma. For $\text{dom } G \subset \mathbb{R} \times (\mathbb{R}^n)^k$ we consider the set of all pairs (t, w) such that $(t_0, w_0) \in \text{dom } f$ and $t \in \text{dom } x$ with x being the maximal solution of the ordinary differential equation (7.1) satisfying the Cauchy conditions (7.2) where $w \in U$. For $\text{cod } G$ take \mathbb{R}^n and set

$$(7.3) \quad G(t, w) = x(t).$$

Clearly, $(t_0, w_0) \in \text{dom } G$. The variable $y: \text{dom } x \ni t \mapsto (x^{(0)}(t), x^{(1)}(t), \dots, x^{(k-1)}(t)) \in (\mathbb{R}^n)^k$ satisfies the first order ordinary differential equation $dy(t)/dt = \Psi(t, y(t))$, where Ψ is the smooth map $(t, (a_0, a_1, \dots, a_{k-1})) \mapsto (a_1, a_2, \dots, a_{k-1}, f(t, (a_0, a_1, \dots, a_{k-1})))$ defined on the open set $\text{dom } \Psi = \text{dom } f$. Every maximal solution of this differential equation satisfying the initial condition $y(t_0) = y_0$ is in the form $y: t \mapsto \Phi(t, t_0, y_0)$, where Φ is a continuous map defined on an open set $\text{dom } \Phi \subset \mathbb{R} \times \text{dom } f$ (see Theorem 14, 23 in [13]). Since the map Φ is

also smooth (see Corollary 4, 7 in [2]) and since $G(t, w) = p(\Phi(t, t_0, w))$, where $p: (\mathbb{R}^n)^k \ni (a_0, a_1, \dots, a_{k-1}) \mapsto a_0 \in \mathbb{R}^n$ denotes the Cartesian projection on the zero factor, then also G is a smooth map defined on an open set $\text{dom } G = \{(t, w) | (t, t_0, w) \in \text{dom } \Phi\} \subset \mathbb{R} \times (\mathbb{R}^n)^k$. If we calculate the value of the Jacobian following the guidelines from the Lemma we get $J(t_0, w_0) = 1$, hence all the requirements of the Lemma are fulfilled.

In this case the conclusions of the Lemma are accomplished too, i.e. there exist such an open interval $I \subset \mathbb{R}$ and such open sets $U \subset (\mathbb{R}^n)^k$ that $(t_0, w_0) \in I \times U \subset \text{dom } G \subset \text{dom } f$ and there is a k -frontal embedding ω given by (6.1). By (7.3) the set $\text{cod } \omega$ is the set of all solutions x of the ordinary differential equation (7.1) defined on the interval I and satisfying the Cauchy conditions (7.2) with $w \in U$.

By Theorem 2, the map φ defined in (5.2) is a solution of functional equations (3.1). By Theorem 1, φ is indeed a flow with limited intersection of worldlines. The definition of flow with limited intersection of worldlines says that the set $X = \text{cod } \varphi$ is the set of worldlines satisfying the condition (2.1). The definition of the set of worldlines with limited intersection then identifies the set X as such a set. Finally, (5.2) gives that $X = \text{cod } \varphi = \text{cod } \omega$ and this fact finishes the proof. \square

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